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MORE ABOUT CLASSICAL VISUAL GROUPS

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Abstract. This paper is devoted to a new perspective to classical visual groups and their fundamental invariants. It proposes a new approach to determine visual intrinsic invariants, visual Frenet-Serret frames and the associated formulas, visual angles and drawing visual normals. Moreover, it introduces Visual Structural Theorem (VST) −indicating that each regular-smooth curve in a visual geometry satisfies a unique third order differential equation, and offers its inverse as a conjecture.

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1 Introduction

Geometric invariants play a key role in object recognition where the object of interest is affected by a group of transformations. A modern approach to them was taken by Weiss (1993); Bruckstein & Shaked (1998); Olver (1999); Aghayan et al. (2014), and Aghayan (2021a,b, 2018, 2017, 2015); Nadjafikhah & Aghayan (2012), etc.

In visual applications, the transformation group $\mathbb{G}$ is, typically, either the Euclidean, affine, similarity, or projective, called visual groups. We are particularly interested in how the geometry induced by $\mathbb{G}$[1] applies to curves contained in the plane $\mathrm{E} \simeq \mathbb{R}^2$. This paper undertakes a systematic investigation of these groups and their fundamental invariants.

This paper is organized as follows. Section 2 presents our approach to determine the intrinsic invariants associated with a geometric property - in our study Length, Angle, and Area, and derives the visual curvatures. We also discuss the relation between convexity and regularity. Section 3 starts with the well-known Euclidean Frenet-Serret frame and formulas, then, gives the associated moving frames and formulas in the similarity and affine geometries. For each visual group, we describe the derivatives of a regular smooth curve in the TN frame to derive the associated fundamental invariants in terms of the inner and cross products. Besides, we bring forward a geometric interpretation by presenting the visual angle in each geometry and showing that this angle determines the normal direction in that geometry. This section ends with expressing the visual curvatures in terms of the tangential angles. Section 4 introduces "visual Structural equations", indicating that any regular smooth curve in a visual geometry satisfies a unique differential equation of order 3, and offers its converse as a conjecture.

[1]in Euclidean, affine, and similarity cases. The same process could be accomplished for Projective group.

2 Visual-Geometric Features and Intrinsic Invariants

According to Olver (1999), any r-dimensional visual group $\mathbb{G}$ acting on the space $\mathrm{E} \simeq \mathbb{R}^2$ admits a unique (up to function thereof) differential invariant $\kappa_{\mathbb{G}}$ of order r−1, called the $\mathbb{G}$-curvature, and a unique (up to constant multiple) invariant one-form $d\iota_{\mathbb{G}}$ of order at most r−2, called the $\mathbb{G}$-arc length. Besides, every other differential invariant is a function $\mathrm{I} = \mathrm{I}(\kappa_{\mathbb{G}}, \kappa'_{\mathbb{G},\iota}, \kappa''_{\mathbb{G},\iota}, \ldots)$ of the $\mathbb{G}$-curvature and its derivatives with respect to the $\mathbb{G}$-arc length element.

As mentioned above, determining *the fundamental invariants associated with a geometric property of interest* plays a central role in recognition problems and finding symmetries. Our method to derive them involves the following steps.

Let $\gamma : [\mathrm{a}, \mathrm{b}] \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ be a regular and s-smooth [1] curve.

Step 1. Determining the arc length $d\iota$ associated with 'the geometric feature of interest' χ - in our study Length, Angle, and Area.

Step 2. Finding two vectors $\upsilon(\gamma'_\iota, \gamma''_\iota, \ldots)$ and $\omega(\gamma'_\iota, \gamma''_\iota, \ldots)$ in terms of the derivatives of the given curve with respect to the arc length ι such that the value of the feature of interest is 1 - in other words, $\chi(\upsilon, \omega) = 1$.

It is obvious that the method presents the simplest invariant if the vectors υ and ω are (as much as possible) in terms of the lowest derivatives.

Step 3. Finally, taking the derivative $\dfrac{d}{d\iota}(\chi(\upsilon, \omega)) = 0$ results in the intrinsic curvature.

In the following, we apply our method to derive the intrinsic curvatures associated with Length, Angle, and Area.

- Length

Let $\gamma = \mathrm{x}\ \hat{\imath} + \mathrm{y}\ \hat{\jmath}$ and $\chi(\mathrm{v}, \mathrm{w}) = \mathrm{v} \cdot \mathrm{w}$ denote the inner product of two vectors v and w.

Step 1. Since $\|\mathrm{v}\| = \sqrt{\mathrm{v} \cdot \mathrm{v}}$ for the Euclidean norm $\|\ \|$, we consider

$$d\mathrm{s} = \|d\gamma\| = \sqrt{d\mathrm{x}^2 + d\mathrm{y}^2} \tag{1}$$

as the arc length for this case.

Step 2. Hence, assuming $\upsilon = \omega = \frac{d\gamma}{d\mathrm{s}}$ gives

$$\chi(\upsilon, \upsilon) = \|\frac{d\gamma}{d\mathrm{s}}\| = 1. \tag{2}$$

Step 3. Finally, taking the derivative on both sides results in

$$\frac{d^2\gamma}{d\mathrm{s}^2} = \kappa\ \frac{d\gamma}{d\mathrm{s}}. \tag{3}$$

It is easy to check that

$$\kappa(\mathrm{p}) = \frac{d\phi}{d\mathrm{s}}(\mathrm{p}) \tag{4}$$

in which $\phi(\mathrm{s})$ denotes the tangential angle of the curve γ, see FIG. 1.

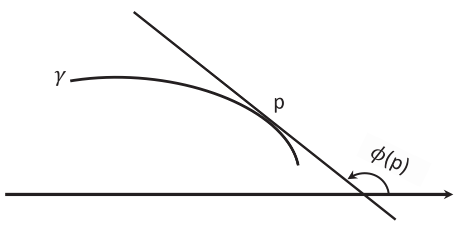

Figure 1: The tangential angel of a smooth curve γ at p.

Moreover, the rate of change in the direction of the tangent line with respect to "s" is clearly the simplest invariant of the Euclidean motions that preserve Length.

Note also that, the curve γ must be regular and 2-smooth to have the invariant curvature κ well-defined.

[1] γ is said to be s-smooth if it is s times differentiable. Also, is regular if its first derivative never vanishes.

- Angle

In this case, let $\chi(\mathrm{v}, \mathrm{w}) = \frac{\mathrm{v} \cdot \mathrm{w}}{\|\mathrm{v}\|\|\mathrm{w}\|}$ denote the angle between two vectors v and w.

Step 1. By reparameterizing the curve γ with the tangential angle ϕ , we consider

$$d\sigma = d\phi \tag{5}$$

as the associated arc length. Therefore, identity (4) gives

$$\frac{d\sigma}{d\mathrm{s}} = \frac{d\phi}{d\mathrm{s}} = \kappa. \tag{6}$$

Step 2. If we consider $\upsilon = \omega = \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}$, then

$$\chi(\upsilon, \upsilon) = \frac{\gamma'_\sigma \cdot \gamma'_\sigma}{\|\gamma'_\sigma\|\|\gamma'_\sigma\|} = \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} \cdot \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} = 1.$$

Step 3. Finally, taking the derivative on both sides results in

$$\frac{\gamma''_\sigma}{\|\gamma'_\sigma\|} \cdot \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} = \frac{1}{\|\gamma'_\sigma\|} \frac{d}{d\sigma}(\|\gamma'_\sigma\|). \tag{7}$$

We define

$$\nu(\sigma) = -\frac{\gamma''_\sigma}{\|\gamma'_\sigma\|} \cdot \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}. \tag{8}$$

Now, we demonstrate that $\nu = \nu(\sigma)$ is the simplest invariant. By (5)

$$\frac{d\phi}{d\sigma} = 1 \Longrightarrow \|\gamma'_\sigma\| \cdot \frac{d\phi}{d\mathrm{s}} = \pm 1.$$

Hence, by taking the derivative on both sides we have

$$\begin{aligned} \frac{d}{d\sigma}(\|\gamma'_\sigma\| \cdot \frac{d\phi}{d\mathrm{s}}) = 0 \quad &\Longrightarrow \quad \frac{d}{d\sigma}(\|\gamma'_\sigma\|) \cdot \phi_\mathrm{s} = -\|\gamma'_\sigma\| \cdot \frac{d}{d\sigma}(\phi_\mathrm{s}) \\ &\Longrightarrow \quad \frac{1}{\|\gamma'_\sigma\|} \cdot \frac{d}{d\sigma}(\|\gamma'_\sigma\|) = \frac{d}{d\mathrm{s}}(\frac{1}{\phi_\mathrm{s}}) \end{aligned}$$

which in fact, according to (7) and (8), shows that

$$\nu = \frac{\kappa_\mathrm{s}}{\kappa^2}. \tag{9}$$

Clearly, this ratio is the simplest invariant of Angle preserving (similarity) transformations

$$\{\mathbf{x} \longmapsto \lambda\mathbf{x} + \mathbf{a} \mid \lambda > 0,\ \mathbf{x}, \mathbf{a} \in \mathbb{R}^2\}.$$

Note further that, according to (8), γ must be a regular 2-smooth convex (no three points on γ are collinear) curve to have the invariant ν well-defined.

- Area

For this case, let $\chi(\mathrm{v}, \mathrm{w}) = \mathrm{v} \times \mathrm{w} = |\mathrm{v}\ \ \mathrm{w}|$ be the determinant of two vectors v and w.

To determine the arc length in this case, we consider a coordinate system for the plane so that the area of the parallelogram spanned by any pair of vectors v and w equals $\chi(\mathrm{v}, \mathrm{w})$ and particularly the determinant $\chi(\gamma'_\mathrm{t}, \gamma''_\mathrm{t})$ gives the signed area of the parallelogram spanned by the velocity and the acceleration of the curve γ.

Now, let "α" denote a reparameterization such that $\gamma = \gamma(\alpha)$ satisfies

$$\chi(\gamma'_\alpha, \gamma''_\alpha) = \gamma'_\alpha \times \gamma''_\alpha = 1 \tag{10}$$

which results in

$$\frac{d\alpha}{d\mathrm{s}} = \kappa^{1/3}. \tag{11}$$

Taking the derivative of (10) results in $\gamma'_\alpha \times \gamma'''_\alpha = 0 \Rightarrow \gamma'''_\alpha + \mu\ \gamma'_\alpha = 0$. Or equivalently

$$\mu = \gamma''_\alpha \times \gamma'''_\alpha. \tag{12}$$

Section 3 will show $\gamma''_\alpha \times \gamma'''_\alpha$ is the simplest non-constant area spanned by the derivatives of γ, in consequence, μ is the simplest invariant of the area preserving affine transformations $\mathrm{x} \longmapsto \mathrm{Ax} + \mathrm{b}$, where $\mathrm{A} \in \mathrm{SL}(2)$ and $\mathrm{x}, \mathrm{b} \in \mathbb{R}^2$. Besides, γ must be a regular 3-smooth convex curve to have the intrinsic invariant μ well-defined.

Remark 1. *Convexity & Regularity. According to (6) and (11)*

$$\frac{d\gamma}{d\sigma} = \frac{ds}{d\sigma}\frac{d\gamma}{ds} \implies \frac{d\gamma}{d\sigma} = \frac{1}{\kappa}\frac{d\gamma}{ds} \qquad \textit{and}$$
$$\frac{d\gamma}{d\alpha} = \frac{dt}{d\alpha}\frac{d\gamma}{dt} \implies \frac{d\gamma}{d\sigma} = \frac{1}{\kappa^{1/3}}\frac{d\gamma}{ds}$$

which demonstrate that a regular curve γ in Euclidean geometry is also regular in similarity and affine geometries, if its Euclidean curvature κ never vanishes, and vice versa.

We, therefore, have the following theorem.

Proposition 1. *A curve γ is regular in similarity and affine geometries if and only if γ is regular and convex in Euclidean geometry.*

Accordingly, from now on, "regular" shall refer to "regular in the geometry in question".

3 Frenet-Serret Frames & Frenet-Serret Formulas

One of the important tools applied to analyze a curve is the *Frenet-Serret frame*, a moving frame that provides a coordinate system at any point of the curve that is "best adapted" to the curve near that point and describes the geometry of the curve at a point completely.
Before studying the Frenet-Serret frame associated with a visual group, we define "the TN frame" built by the following moving vectors along a regular 2-smooth curve $\gamma = \gamma(t)$:

- T is the unit vector tangent to the curve, pointing in the direction of motion: $T(t) = \frac{\gamma_t^{'}}{\|\gamma_t^{'}\|}$.
- N is the normal unit vector, the derivative of T with respect to the arclength parameter of the curve, divided by its length: $N(t) = \frac{T_t^{'}}{\|\gamma_t^{'}\|}$.

Moreover, the *Frenet-Serret formulas* are vector differential equations that relate inherent properties of the curve γ. They describe in fact the derivatives of the unit tangent and normal vectors in terms of each other in which the coefficients are in terms of the intrinsic invariants.

3.1 Euclidean Frenet-Serret frame

Let $\gamma = \gamma(s)$ be a regular curve with respect to the Euclidean arc length s. Since $\|\frac{d\gamma}{ds}\| = 1$, therefore, $T = \frac{d\gamma}{ds}$ is the unit tangent vector. Then

$$\|T\| = 1 \implies \frac{dT}{ds} \cdot T = 0 \implies \|\frac{dT}{ds}\| \cos\theta = 0 \tag{13}$$

which results if γ is not a line, then

$$\frac{dT}{ds} \perp T \implies \frac{dT}{ds} = \kappa \, N. \tag{14}$$

Hence, for a 2-regular curve γ the Euclidean Frenet-Serret frame is the same TN frame:

$$T = \frac{d\gamma}{ds} \quad \text{and} \quad N = \frac{dT}{ds}. \tag{15}$$

Moreover, the associated formulae are given by:

$$\frac{dT}{ds} = \kappa N \quad \text{and} \quad \frac{dN}{ds} = -\kappa T. \tag{16}$$

Unfortunately, this frame is not well-defined in other visual geometries. In this section we introduce a suitable Frenet-Serret frame for the scaling and affine transformations.

3.2 Similarity Frenet-Serret frame

Let $\gamma = \gamma(\sigma)$ be a smooth curve parameterized by the similarity arc length σ. According to (5)-(8)[1]

$$\frac{d\sigma}{ds} = \frac{d\phi}{ds} = \kappa, \quad \|\gamma'_\sigma\| = \kappa^{-1}, \quad \text{and} \quad \nu = -\frac{1}{\|\gamma'_\sigma\|}\,\frac{d}{d\sigma}(\|\gamma'_\sigma\|). \tag{17}$$

Moreover, in terms of the TN frame, we have

$$\mathrm{T}'_\sigma = \mathrm{N} \quad \text{and} \quad \mathrm{N}'_\sigma = -\mathrm{T}. \tag{18}$$

Now, let $\gamma = \gamma(\sigma)$ denote a regular 3-smooth curve. Then

$$\begin{aligned}
\gamma'_\sigma &= \|\gamma'_\sigma\|\ \mathrm{T} = \kappa^{-1}\ \mathrm{T}, &(19)\\
\gamma''_\sigma &= \|\gamma'_\sigma\|'\ \mathrm{T} + \|\gamma'_\sigma\|\ \mathrm{N} = -\kappa^{-3}\kappa'_{\mathrm{s}}\ \mathrm{T} + \kappa^{-1}\ \mathrm{N}, &(20)\\
\gamma'''_\sigma &= (\|\gamma'_\sigma\|'' - \|\gamma'_\sigma\|)\ \mathrm{T} + 2\|\gamma'_\sigma\|'\ \mathrm{N} = (-\kappa^{-4}\kappa''_{\mathrm{s}} + 3\kappa^{-5}\kappa'^{,2}_{\mathrm{s}} - \kappa^{-1})\ \mathrm{T} - 2\kappa^{-3}\kappa'_{\mathrm{s}}\ \mathrm{N}. &(21)
\end{aligned}$$

and

$$\begin{aligned}
\gamma'_\sigma \cdot \gamma'_\sigma &= \|\gamma'_\sigma\|^2 = \kappa^{-2}, &(22)\\
\gamma'_\sigma \cdot \gamma''_\sigma &= -\nu\ \|\gamma'_\sigma\|^2 = -\kappa^{-4}\kappa'_{\mathrm{s}}, &(23)\\
\gamma'_\sigma \cdot \gamma'''_\sigma &= \|\gamma'_\sigma\|(\|\gamma'_\sigma\|'' - \|\gamma'_\sigma\|) = -\kappa^{-5}\kappa''_{\mathrm{s}} + 3\kappa^{-6}\kappa'^{,2}_{\mathrm{s}} - \kappa^{-2}. &(24)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\gamma'_\sigma \times \gamma''_\sigma &= \|\gamma'_\sigma\|^2 = \kappa^{-2}, &(25)\\
\gamma'_\sigma \times \gamma'''_\sigma &= -2\nu\ \|\gamma'_\sigma\|^2 = -2\kappa^{-4}\kappa'_{\mathrm{s}}, &(26)\\
\gamma''_\sigma \times \gamma'''_\sigma &= (\nu^2 - \nu')\|\gamma'_\sigma\|^2 = \kappa^{-6}\kappa''_{\mathrm{s}} + 3\kappa^{-6}\kappa'^{,2}_{\mathrm{s}} + \kappa^{-3}. &(27)
\end{aligned}$$

Therefore, the following results are established. According to (19), (20), and (21)

$$\text{a)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} = \mathrm{T}, \quad \text{b)}\ \frac{\gamma''_\sigma}{\|\gamma'_\sigma\|} = -\nu\ \mathrm{T} + \mathrm{N}, \quad \text{and} \quad \text{c)}\ \frac{\gamma'''_\sigma}{\|\gamma'_\sigma\|} = (-\nu'_\sigma + \nu^2 - 1)\ \mathrm{T} - 2\nu\ \mathrm{N}, \tag{28}$$

which are *similarity invariant vectors*. In addition, identities (22)-(27) result in

$$\text{a)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}\cdot\frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} = 1, \quad \text{b)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}\cdot\frac{\gamma''_\sigma}{\|\gamma'_\sigma\|} = -\nu, \quad \text{and} \quad \text{c)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}\cdot \text{a)}\ \frac{\gamma'''_\sigma}{\|\gamma'_\sigma\|} = -\nu'_\sigma + \nu^2 - 1, \tag{29}$$

$$\text{a)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}\times\frac{\gamma''_\sigma}{\|\gamma'_\sigma\|} = 1, \quad \text{b)}\ \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|}\times\frac{\gamma'''_\sigma}{\|\gamma'_\sigma\|} = -2\nu, \quad \text{and} \quad \text{c)}\ \frac{\gamma''_\sigma}{\|\gamma'_\sigma\|}\times\frac{\gamma'''_\sigma}{\|\gamma'_\sigma\|} = \nu'_\sigma + \nu^2 + 1, \tag{30}$$

which are *similarity invariant functions.*

Theorem 1. *Let $\gamma = \gamma(\sigma)$ denote a regular 2-smooth curve. Then $\{\hat{\mathrm{t}} = \hat{\mathrm{t}}(\sigma), \hat{\mathrm{n}} = \hat{\mathrm{n}}(\sigma)\}$ given by*

$$\hat{\mathrm{t}} = \frac{\gamma'_\sigma}{\|\gamma'_\sigma\|} \quad \textit{and} \quad \hat{\mathrm{n}} = \frac{\gamma''_\sigma}{\|\gamma'_\sigma\|}. \tag{31}$$

defines a moving frame of the curve γ.

[1]From now on, to avoid the ambiguity caused by sign and without loss of generality, we assume κ is positive.

Proof. Let

$$f_1 \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} + f_2 \; \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|} = g_1 \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} + g_2 \; \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|}. \tag{32}$$

Multiplying both sides of (30)-part a) by $\frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|}$

$$f_1 \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} \times \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} + f_2 \; \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|} \times \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} = g_1 \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} \times \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} + g_2 \; \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|} \times \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|}. \tag{33}$$

results in $f_2 = g_2$. Similarly, we have $f_1 = g_1$.

We call $\hat{\mathrm{t}}$ "the similarity tangent", $\hat{\mathrm{n}}$ "the similarity normal", and the moving fame $\{\hat{\mathrm{t}}, \hat{\mathrm{n}}\}$ the "similarity Frenet-Serret frame". For example, taking derivative on both sides of identity (29)-part b) gives the following identity in the $\hat{\mathrm{t}}$-$\hat{\mathrm{n}}$ coordinate plane

$$\frac{\gamma_\sigma^{'''}}{\|\gamma_\sigma^{'}\|} = (-\nu_\sigma^{'} - \nu^2 - 1) \; \hat{\mathrm{t}} - 2\nu \; \hat{\mathrm{n}}. \tag{34}$$

□

It is not difficult to prove the following lemma.

Lemma 1. *Let $\gamma = \gamma(\sigma)$ be a regular smooth curve. Then*

$$\left(\frac{\gamma_\sigma^{(\mathrm{n})}}{\|\gamma_\sigma^{'}\|}\right)_\sigma^{'} = \frac{\gamma_\sigma^{(\mathrm{n}+1)}}{\|\gamma_\sigma^{'}\|} + \nu \; \frac{\gamma_\sigma^{(\mathrm{n})}}{\|\gamma_\sigma^{'}\|}. \tag{35}$$

Theorem 2. Let $\gamma = \gamma(\sigma)$ be a regular 3-smooth curve. Then

$$\mathrm{t}_\sigma^{'} = \nu \; \hat{\mathrm{t}} + \hat{\mathrm{n}} \quad \textit{and} \quad \mathrm{n}_\sigma^{'} = (-\nu_\sigma^{'} - \nu^2 - 1) \; \hat{\mathrm{t}} - \nu \; \hat{\mathrm{n}}. \tag{36}$$

Proof. First, by (35)

$$\left(\frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|}\right)_\sigma^{'} = \nu \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} + \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|} \tag{37}$$

which proves (36). Also, by (29)

$$\left(\frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|}\right)_\sigma^{'} = (-\nu \; \mathrm{T} + \mathrm{N})_\sigma^{'} = (-\nu_\sigma^{'} - 1) \; \mathrm{T} - \nu \; \mathrm{N} = (-\nu_\sigma^{'} - \nu^2 - 1) \; \frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} - \nu \; \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|}.$$

□

We call (36) and (37) (or equivalently, (35)) "the similarity Frenet-Serret formulas".

3.2.1 Similarity angle & geometric interpretation

By (29) and (30), it is easy to check that

$$\begin{aligned}
\cos(\gamma_\sigma^{'}, \gamma_\sigma^{''}) &= \frac{-\nu}{\sqrt{\nu^2+1}}, \quad \cos(\gamma_\sigma^{'}, \gamma_\sigma^{'''}) = \frac{\nu^2 - \nu_\sigma^{'} - 1}{\sqrt{(\nu^2 - \nu_\sigma^{'} + 1)^2 + 4\nu_\sigma^{'}}}, \quad \text{and} \\
\cos(\gamma_\sigma^{''}, \gamma_\sigma^{'''}) &= \frac{-\nu(\nu^2 - \nu_\sigma^{'} + 1)}{\sqrt{\nu^2+1}\sqrt{(\nu^2 - \nu_\sigma^{'} + 1)^2 + 4\nu_\sigma^{'}}}.
\end{aligned} \tag{38}$$

$$\begin{aligned}
\sin(\gamma_\sigma^{'}, \gamma_\sigma^{''}) &= \frac{1}{\sqrt{\nu^2+1}}, \quad \sin(\gamma_\sigma^{'}, \gamma_\sigma^{'''}) = \frac{-2\nu^2}{\sqrt{(\nu^2 - \nu_\sigma^{'} + 1)^2 + 4\nu_\sigma^{'}}}, \quad \text{and} \\
\sin(\gamma_\sigma^{''}, \gamma_\sigma^{'''}) &= \frac{\nu^2 + \nu_\sigma^{'} + 1}{\sqrt{\nu^2+1}\sqrt{(\nu^2 - \nu_\sigma^{'} + 1)^2 + 4\nu_\sigma^{'}}}.
\end{aligned} \tag{39}$$

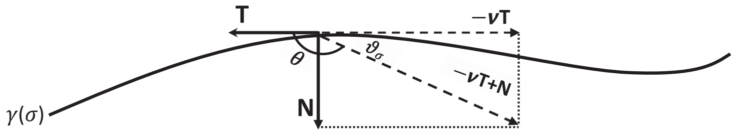

Figure 2: Sketches $\|\gamma_\sigma^{'}\|^{-1}\gamma_\sigma^{''}$ in the TN frame.

We name these similarity invariants "the fundamental similarity angles".

Now, let $\theta = \angle(\gamma_\sigma^{'}, \gamma_\sigma^{''})$ be the angle between $\gamma_\sigma^{'}$ and $\gamma_\sigma^{''}$. Identities (28) and (29)

$$\frac{\gamma_\sigma^{'}}{\|\gamma_\sigma^{'}\|} = \mathrm{T} \quad \text{and} \quad \frac{\gamma_\sigma^{''}}{\|\gamma_\sigma^{'}\|} = -\nu\ \mathrm{T} + \mathrm{N}, \tag{40}$$

with regard to Figure 2, result in: $\cot(\theta) = -\cot(\vartheta_\sigma = \pi - \theta) = -\nu \Longleftrightarrow \vartheta_\sigma = \cot^{-1}(\nu)$.

We, therefore, have the following theorem.

Theorem 3. *Let $\{\hat{\mathrm{t}}, \hat{\mathrm{n}}\}$ be similarity Frenet-Serret frame of a regular 3-smooth curve $\gamma = \gamma(\sigma)$. Let $\vartheta_\sigma = \angle(-\gamma_\sigma^{'}, \gamma_\sigma^{''}) = \angle(-\hat{\mathrm{t}}, \hat{\mathrm{n}})$. Then*

$$\vartheta_\sigma = \cot^{-1}(\nu) \tag{41}$$

where $\nu = \nu(\sigma)$ denotes the similarity curvature of γ.

We call ϑ_σ "the similarity angle" of γ. It is also easy to check that the angle ϑ_σ at a point $\mathrm{p} \in \gamma$ is the half of the central angle in the osculating circle C at this point whose vertices are the touching point p and the intersection of the similarity normal with C, see Fig. 3.

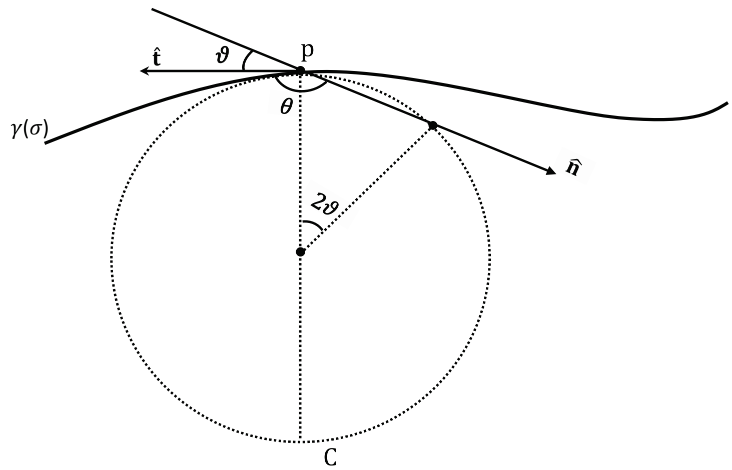

Figure 3: The similarity angle of a regular smooth curve $\gamma(\sigma)$ and its geometrical interpretation.

3.3 Affine Frenet-Serret frame

Let γ denote a regular 3-smooth curve with the Euclidean, similarity, and affine curvatures κ, ν, and μ. Let s, σ, and α be respectively the Euclidean, similarity, and affine arc lengths. It is not difficult to check that:

$$\frac{d\alpha}{d\mathrm{s}} = \kappa^{1/3} = \frac{1}{\|\gamma_\alpha^{'}\|} \quad \text{or} \quad \kappa = \|\gamma_\alpha^{'}\|^{-3}, \quad \text{and} \quad \frac{d\alpha}{d\sigma} = \kappa^{-2/3} = \|\gamma_\alpha^{'}\|^2, \tag{42}$$

$$\mathrm{T}_\alpha^{'} = \|\gamma_\alpha^{'}\|^{-2}\ \mathrm{N} \quad \text{and} \quad \mathrm{N}_\alpha^{'} = -\|\gamma_\alpha^{'}\|^{-2}\ \mathrm{T} \quad \text{in the TN frame of } \gamma, \tag{43}$$

$$\|\gamma_\alpha^{'}\|_\alpha^{'} = -\frac{\kappa^{1/3}\nu}{3} \quad \text{and} \quad \|\gamma_\alpha^{'}\|_\alpha^{''} = \kappa(-\frac{\nu_\sigma^{'}}{3} - \frac{\nu^2}{9}), \tag{44}$$

which prove the following identities.

$$\gamma'_\alpha = \|\gamma'_\alpha\|\ \mathrm{T} = \kappa^{-1/3}\ \mathrm{T}, \tag{45}$$

$$\gamma''_\alpha = \|\gamma'_\alpha\|'_\alpha\ \mathrm{T} + \|\gamma'_\alpha\|^{-1}\ \mathrm{N} = -\frac{\kappa^{1/3}\nu}{3}\ \mathrm{T} + \kappa^{1/3}\ \mathrm{N} = -\frac{\kappa^{-5/3}\kappa'_s}{3}\ \mathrm{T} + \kappa^{1/3}\ \mathrm{N}, \tag{46}$$

$$\gamma'''_\alpha = (\|\gamma'_\alpha\|''_\alpha - \|\gamma'_\alpha\|^{-3})\ \mathrm{T} = (-\frac{\kappa\nu'_\sigma}{3} - \frac{\kappa\nu^2}{9} - \kappa)\ \mathrm{T} = (-\frac{\kappa^{-2}\kappa''_s}{3} + \frac{5\kappa^{-3}\kappa'^{,2}_s}{9} - \kappa)\ \mathrm{T}. \tag{47}$$

Then

$$\gamma'_\alpha \cdot \gamma'_\alpha = \|\gamma'_\alpha\|^2 = \kappa^{-2/3}, \tag{48}$$

$$\gamma'_\alpha \cdot \gamma''_\alpha = \|\gamma'_\alpha\|\|\gamma'_\alpha\|'_\alpha = -\frac{\nu}{3} = -\frac{\kappa^{-2}\kappa'_s}{3}, \tag{49}$$

$$\gamma'_\alpha \cdot \gamma'''_\alpha = \|\gamma'_\alpha\|\|\gamma'_\alpha\|''_\alpha - \|\gamma'_\alpha\|^{-2} \tag{50}$$

$$= -\kappa^{2/3}(\frac{\nu'_\sigma}{3} + \frac{\nu^2}{9} + 1) = -\frac{\kappa^{-7/3}\kappa''_s}{3} + \frac{5\kappa^{-10/3}\kappa'^{,2}_s}{9} - \kappa^{2/3}, \tag{51}$$

$$\gamma'_\alpha \times \gamma''_\alpha = 1, \tag{52}$$

$$\gamma'_\alpha \times \gamma'''_\alpha = 0, \tag{53}$$

$$\gamma''_\alpha \times \gamma'''_\alpha = \frac{\|\gamma'_\alpha\|''_\alpha}{\|\gamma'_\alpha\|} - \frac{1}{\|\gamma'_\alpha\|^4} = \kappa^{4/3}(\frac{\nu'_\sigma}{3} + \frac{\nu^2}{9} + 1) = \frac{\kappa^{-5/3}\kappa''_s}{3} - \frac{5\kappa^{-8/3}\kappa'^{,2}_s}{9} + \kappa^{4/3}. \tag{54}$$

Now, we make some results. Identities (45), (46), and (47) give the following corollary.

Corollary 1. Let $\gamma = \gamma(\alpha)$ denote a regular 3-smooth curve. Then

- $\|\gamma'_\alpha\|$, $\|\gamma''_\alpha\|$, *and* $\|\gamma'''_\alpha\|$ *are Euclidean invariants.*
- $\|\gamma'_\alpha\|\|\gamma''_\alpha\|$, *and* $\|\gamma'_\alpha\|^{-3}\|\gamma'''_\alpha\|$ *are similarity invariants.*

Also, identities (48) and (49) give the outcome below.

Corollary 2. Let κ and ν denote, respectively, the Euclidean and similarity curvatures of a regular 3-smooth curve $\gamma = \gamma(\alpha)$. Then

$$\kappa = (\gamma'_\alpha \cdot \gamma'_\alpha)^{-3/2}, \tag{55}$$

$$\nu = -\frac{1}{3}(\gamma'_\alpha \cdot \gamma''_\alpha). \tag{56}$$

These identities, indeed, generate the Euclidean and similarity curvatures in affine geometry. In addition, identity (54), along with (12), demonstrates the following result.

Corollary 3. *Let $\gamma = \gamma(\alpha)$ denote a regular 3-smooth curve. Then*

$$\mu = \frac{\|\gamma'_\alpha\|^3\|\gamma'_\alpha\|''_\alpha - 1}{\|\gamma'_\alpha\|^4} = \frac{\frac{1}{3}\nu'_\sigma + \frac{1}{9}\nu^2 + 1}{\|\gamma'_\sigma\|^{4/3}} = \frac{3\kappa\kappa''_s - 5\kappa'^{,2}_s + 9\kappa^4}{9\kappa^{8/3}}. \tag{57}$$

Now, we bring forward the affine Frenet-Serret frames.

Theorem 4. *Let $\gamma(\alpha)$ be a 2-smooth convex curve. Then, $\{\hat{\mathbb{T}} = \hat{\mathbb{T}}(\alpha), \hat{\mathbb{N}} = \hat{\mathbb{N}}(\alpha)\}$, given as follows, defines a moving frame over the curve γ.*

$$\hat{\mathbb{T}} = \gamma'_\alpha \quad \text{and} \quad \hat{\mathbb{N}} = \gamma''_\alpha. \tag{58}$$

Proof. Let

$$f_1\ \gamma'_\alpha + f_2\ \gamma''_\alpha = g_1\ \gamma'_\alpha + g_2\ \gamma''_\alpha. \tag{59}$$

With regard to (54), multiplying both sides by γ'_α

$$f_1\ \gamma'_\alpha \times \gamma'_\alpha + f_2\ \gamma'_\alpha \times \gamma''_\alpha = g_1\ \gamma'_\alpha \times \gamma'_\alpha + g_2\ \gamma'_\alpha \times \gamma''_\alpha \tag{60}$$

gives $f_1 = g_1$. Similarly, we have $f_2 = g_2$.

The moving fame $\{\hat{\mathbb{T}}, \hat{\mathbb{N}}\}$ is called "the affine Frenet-Serret frame" of the curve γ. Besides, $\hat{\mathbb{T}}$ and $\hat{\mathbb{N}}$ are respectively named its "affine tangent" and "affine normal". ☐

Moreover, it is not difficult to establish the theorem below.

Theorem 5. *Let $\gamma = \gamma(\alpha)$ be a 3-smooth convex curve. Then*

$$\hat{\mathbb{T}}_{\alpha}' = \hat{\mathbb{N}} \quad \text{and} \quad \hat{\mathbb{N}}_{\alpha}' = -\mu\hat{\mathbb{T}}. \tag{61}$$

These equations are called "the affine Frenet-Serret formula" of γ.

3.3.1 Affine angle & geometric interpretation

Identity (59) results in

$$\|\gamma_{\alpha}'\|\gamma_{\alpha}'' = -\frac{1}{3}\nu \ \mathrm{T} + \mathrm{N}.$$

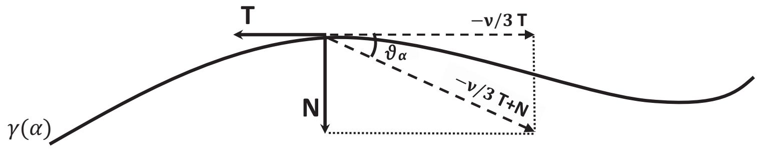

Figure 4: The affine angle of a regualr smooth curve $\gamma(\alpha)$.

Hence, according to Figure 4, the angle $\vartheta_{\alpha} = \cot^{-1}(\nu/3)$ points out the direction of the affine normal of γ - named "the affine angle" of the curve γ.

The following figure sketches the Euclidean, similarity, and affine angles and their normal directions. In fact, *each visual angle refers to the angle in the associated Frenet-Serret frame.* Additionally, the interesting point is the similarity and affine angles are both in terms of the similarity curvature of the given curve.

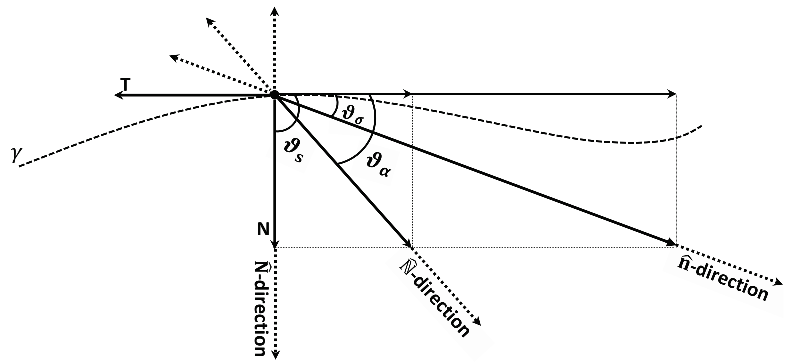

Figure 5: The visual angles and normals for a regular smooth curve γ.

3.4 Visual Curvatures in terms of the Tangential Angles

This section writes the visual curvatures in terms of the Euclidean tangential angles $\phi(\mathrm{s})$, see Fig. 1. For the Euclidean and similarity curvatures, with regard to (4) and (9), we have

$$\kappa = \frac{d\phi}{ds} \quad \text{and} \quad \nu = \frac{\kappa}{\kappa_{\mathrm{s}}} = -\frac{d}{ds}(\frac{1}{\phi_{\mathrm{s}}'}).$$

For the affine case, by (57)

$$\mu = \frac{(\frac{dr}{ds})^2 - 3r\frac{d^2r}{ds^2} - 9}{9r^{4/3}}. \tag{62}$$

in which $r(\mathrm{s})$ denotes the curvature radius of the smooth convex curve γ. On the other hand

$$\frac{(\frac{dr}{ds})^2 - 3r\frac{d^2r}{ds^2}}{9r^{4/3}} = -\frac{1}{2}\frac{d^2}{ds^2}(r^{2/3}) \tag{63}$$

Accordingly, substituting (63) in (62) results in

$$\mu = -\frac{1}{2}\frac{d^2}{ds^2}(r^{2/3}) + \frac{1}{r^{4/3}}. \tag{64}$$

We, therefore, established the following theorem.

Theorem 6. *Let $\phi = \phi(s)$ be the Euclidean tangential angle of a smooth convex curve γ. Let κ, ν, and, μ be the Euclidean, similarity and affine curvatures of γ. Then*

$$\kappa = \frac{d}{ds}(\phi), \quad \nu = -\frac{d}{ds}\left(\frac{1}{\phi_s'}\right), \quad \textit{and} \quad \mu = -\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{\phi_s'}\right)^{2/3} + \phi_s^{',4/3}. \tag{65}$$

Now, we write the affine curvature μ in terms of the tangential angles $\phi(\alpha)$ as a function of the affine arc length. It is easy to check that

$$\frac{d\alpha}{ds} = \phi_\alpha^{',1/2} \quad \text{and} \quad \kappa = \phi_\alpha^{',3/2}. \tag{66}$$

As a result

$$\kappa_s' = \frac{3}{2}\phi_\alpha'\phi_\alpha'' \qquad \text{and} \tag{67}$$

$$\kappa_s'' = \frac{3}{2}\phi_\alpha^{',1/2}\phi_\alpha^{'',2} + \frac{3}{2}\phi_\alpha^{',3/2}\phi_\alpha'''. \tag{68}$$

Replacing (66), (67), and (68) in (57) gives

$$\mu = \frac{-\frac{3}{4}\phi_\alpha^{',2}\phi_\alpha^{'',2} + \frac{1}{2}\phi_\alpha^{',3}\phi_\alpha''' + \phi_\alpha^{',6}}{\phi_\alpha^{',4}}. \tag{69}$$

Therefore, we have the following theorem.

Theorem 7. *Let μ and $\phi = \phi(\alpha)$ denote the affine curvature and the affine tangential angles of a smooth convex curve γ. Then*

$$\mu = \frac{-3\phi_\alpha^{'',2} + 2\phi_\alpha'\phi_\alpha''' + 4\phi_\alpha^{',4}}{4\phi_\alpha^{',2}}. \tag{70}$$

In addition, by (66)

$$\kappa^{2/3}\left(\frac{1}{\phi_\alpha'}\right) = 1. \tag{71}$$

Taking derivative on both sides with respect to the Euclidean arc length results in

$$\frac{2}{3}\kappa_s'\kappa^{-1} + \kappa^{2/3}\frac{d}{ds}\left(\frac{1}{\phi_\alpha'}\right) = 0. \tag{72}$$

By taking the next derivative

$$\frac{2}{3}\kappa_s''\kappa^{-1} - \frac{2}{3}\kappa_s^{',2}\kappa^{-2} + \frac{2}{3}\kappa_s'\kappa^{-1/3}\frac{d}{ds}\left(\frac{1}{\phi_\alpha'}\right) + \kappa^{2/3}\frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) = 0$$

$$\Longrightarrow \quad \frac{2}{3}\kappa_s''\kappa^{-1} - \frac{10}{9}\kappa_s^{',2}\kappa^{-2} + \kappa^{2/3}\frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) = 0 \Longrightarrow \frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) = -\frac{2}{3}\kappa_s''\kappa^{-5/3} + \frac{10}{9}\kappa_s^{',2}\kappa^{-8/3}$$

which gives

$$-\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) + \kappa^{4/3} = \frac{1}{3}\kappa_s''\kappa^{-5/3} - \frac{5}{9}\kappa_s^{',2}\kappa^{-8/3} + \kappa^{4/3}. \tag{73}$$

Hence, according to (57)

$$\mu = -\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) + \kappa^{4/3}. \tag{74}$$

We, therefore, established the following theorem.

Theorem 8. *Let μ and $\phi = \phi(\alpha)$ denote the Affine curvature and the Affine tangential angles of a smooth convex curve γ. Let s refer to the Euclidean arc length element. Then*

$$\mu = -\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{\phi_\alpha'}\right) + \phi_\alpha^{',2}. \tag{75}$$

In fact, what the paper indicates so far is:
"a visual geometry is the structure created by the derivatives of the smooth curves with respect to a given arc length."

The following table gives a summary of the formulae we derived in this paper for the visual curvatures in terms of the dot product, cross product, and norm.

Visual curvatures in terms of inner product, vector product, and norms			
Curvature	in terms of "·"	in terms of "×"	in terms of "‖ ‖"
Euclidean	$-\kappa = \gamma_s' \cdot \gamma_s'''$	$\kappa = \gamma_s' \times \gamma_s'''$	$\lvert\kappa\rvert = \lVert\gamma_s''\rVert$
Similarity	$-\nu = \frac{\gamma_\sigma'}{\lVert\gamma_\sigma'\rVert} \cdot \frac{\gamma_\sigma''}{\lVert\gamma_\sigma'\rVert}$	$-2\nu = \frac{\gamma_\sigma'}{\lVert\gamma_\sigma'\rVert} \times \frac{\gamma_\sigma'''}{\lVert\gamma_\sigma'\rVert}$	$\nu = -\frac{\lVert\gamma_\sigma'\rVert_\sigma'}{\lVert\gamma_\sigma'\rVert}$
Affine	$-\mu = \frac{\gamma_\alpha'}{\lVert\gamma_\alpha'\rVert} \cdot \frac{\gamma_\alpha'''}{\lVert\gamma_\alpha'\rVert}$	$\mu = \gamma_\alpha'' \times \gamma_\alpha'''$	$\lvert\mu\rvert = -\frac{\lVert\gamma_\alpha'''\rVert}{\lVert\gamma_\alpha'\rVert}$

Now, we can have the following definition.

Definition 1. *The infinitesimal $\mathcal{G}$-generators of the visual groups are defined as follows.*
- For Euclidean geometry

$$|\kappa| = \|\gamma_s''\|. \tag{76}$$

- For Similarity geometry

$$\nu = -\frac{\|\gamma_\sigma'\|_\sigma'}{\|\gamma_\sigma'\|}. \tag{77}$$

- For Affine geometry

$$|\mu| = -\frac{\|\gamma_\alpha'''\|}{\|\gamma_\alpha'\|}. \tag{78}$$

In which γ denotes an arbitrary regular smooth curve.

Accordingly,
to have a numerically invariant expression for the infinitesimal generator of a visual geometry one only needs to numerically compute the arc length in a fully group-invariant manner, for example in terms of the joint invariants, and then takes the required derivatives with respect to it.

4 Structural Equations for Visual Groups

This section explores "the structural equation" of each visual group satisfied by all smooth curves in the underlying geometry.

4.1 Structural equation in Euclidean geometry

Let γ be a regular smooth curve and consider its TN frame. From (15) and (16)

$$\gamma_s' = \mathrm{T} \qquad \text{and} \qquad \gamma_s'' = \kappa\ \mathrm{N}. \tag{79}$$

Hence

$$\begin{aligned}\|\kappa^{-1}\gamma_{\mathrm{s}}^{''}\| = 1 \quad &\Longrightarrow \quad (\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'} = \xi_1\ \mathrm{T} \quad \Longrightarrow \quad \xi_1 = \langle (\kappa^{-1})^{'}_{\mathrm{s}}\gamma_{\mathrm{s}}^{''} + \kappa^{-1}\gamma_{\mathrm{s}}^{'''},\ \mathrm{T}\rangle \\ &\Longrightarrow \quad \xi_1 = (\kappa^{-1})^{'}_{\mathrm{s}}\langle \gamma_{\mathrm{s}}^{''},\ \mathrm{T}\rangle + \kappa^{-1}\langle \gamma_{\mathrm{s}}^{'''},\ \mathrm{T}\rangle \quad \Longrightarrow \quad \xi_1 = -\kappa.\end{aligned}$$

Therefore

$$(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'} = -\kappa\ \mathrm{T} \qquad \text{or} \qquad -\kappa^{-1}(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'} = \mathrm{T} \tag{80}$$

which result in

$$(-\kappa^{-1}(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'})^{'} = \xi_2\ \mathrm{N} \quad \Longrightarrow \quad \xi_2 = \langle (\kappa^{-3}\kappa_{\mathrm{s}}^{'}\gamma_{\mathrm{s}}^{''} - \kappa^{-2}\gamma_{\mathrm{s}}^{'''})^{'},\ \mathrm{N}\rangle.$$

Accordingly

$$\begin{aligned}\xi_2 &= (-3\kappa^{-4}\kappa_{\mathrm{s}}^{',2} + \kappa^{-3}\kappa_{\mathrm{s}}^{''})\langle \gamma_{\mathrm{s}}^{''},\ \mathrm{N}\rangle + (\kappa^{-3}\kappa_{\mathrm{s}}^{'} + 2\kappa^{-3}\kappa_{\mathrm{s}}^{'})\langle \gamma_{\mathrm{s}}^{'''},\ \mathrm{N}\rangle - \kappa^{-2}\langle \gamma_{\mathrm{s}}^{''''},\ \mathrm{N}\rangle \\ \Longrightarrow \quad \xi_2 &= -3\kappa^{-3}\kappa_{\mathrm{s}}^{',2} + \kappa^{-2}\kappa_{\mathrm{s}}^{''} + \kappa^{-3}\kappa_{\mathrm{s}}^{',2} + 2\kappa^{-3}\kappa_{\mathrm{s}}^{',2} - \kappa^{-2}\kappa_{\mathrm{s}}^{''} + \kappa \quad \Longrightarrow \quad \xi_2 = \kappa.\end{aligned}$$

Hence

$$-(\kappa^{-1}(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'})^{'} = \kappa\ \mathrm{N} \qquad \text{or} \qquad -\kappa^{-1}(\kappa^{-1}(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'})^{'} = \mathrm{N}. \tag{81}$$

Identity (77) (or equivalently (78)), along with (76), establishes the following theorem.

Theorem 9. ***Euclidean structural equation***. *Any regular and smooth curve $\gamma = \gamma(\mathrm{s})$ satisfies the following third order differential equation*

$$\kappa^{-1}(\kappa^{-1}\gamma_{\mathrm{s}}^{''})^{'} + \gamma_{\mathrm{s}}^{'} = 0 \tag{82}$$

in Euclidean geometry where s denotes the Euclidean arc length.

Conjecture I. The unique planar geometry with an arclength ι in which any regular smooth curve $\gamma = \gamma(\iota)$ is a solution to the the 3$^{\mathrm{rd}}$ ordinary differential equation

$$\kappa^{-1}(\kappa^{-1}\gamma_{\iota}^{''})^{'} + \gamma_{\iota}^{'} = 0 \tag{83}$$

is the Euclidean with $\iota = \mathrm{s}$.

4.2 Structural equation in similarity geometry

With regard to (20)

$$\kappa\gamma_{\sigma}^{'} = \mathrm{T}. \tag{84}$$

Hence

$$\begin{aligned}\|\kappa\gamma_{\sigma}^{'}\| &= 1 \quad \Longrightarrow \quad (\kappa\gamma_{\sigma}^{'})_{\sigma}^{'} = \xi_1\ \mathrm{N} \\ \Longrightarrow \quad \xi_1 &= \langle \kappa_{\sigma}^{'}\gamma_{\sigma}^{'} + \kappa\gamma_{\sigma}^{''},\ \mathrm{N}\rangle = \kappa_{\sigma}^{'}\langle \gamma_{\sigma}^{'},\ \mathrm{N}\rangle + \kappa\langle \gamma_{\sigma}^{''},\ \mathrm{N}\rangle \quad \Longrightarrow \quad \xi_1 = 1.\end{aligned}$$

Therefore

$$(\kappa\gamma_{\sigma}^{'})_{\sigma}^{'} = \mathrm{N} \tag{85}$$

which results in

$$\begin{aligned}(\kappa\gamma_{\sigma}^{'})_{\sigma}^{''} = \xi_2\ \mathrm{T} \quad &\Longrightarrow \quad \xi_2 = \langle (\kappa_{\sigma}^{'}\gamma_{\sigma}^{'} - \kappa\gamma_{\sigma}^{''})_{\sigma}^{'},\ \mathrm{T}\rangle \\ &\Longrightarrow \quad \xi_2 = \kappa_{\sigma}^{''}\langle \gamma_{\sigma}^{'}, \mathrm{T}\rangle + 2\kappa_{\sigma}^{'}\langle \gamma_{\sigma}^{''}, \mathrm{T}\rangle + \kappa\langle \gamma_{\sigma}^{'''}, \mathrm{T}\rangle \quad \Longrightarrow \quad \xi_2 = 1.\end{aligned}$$

Hence

$$(\kappa\gamma_{\sigma}^{'})_{\sigma}^{''} = \mathrm{T}. \tag{86}$$

Continuing the same process gives

$$\frac{d^{2n}}{d\sigma^{2n}}(\kappa\gamma'_\sigma) = \mathrm{T} \qquad \text{and} \qquad \frac{d^{2n+1}}{d\sigma^{2n+1}}(\kappa\gamma'_\sigma) = \mathrm{N} \quad \text{for all} \quad n \in \mathbb{N}. \tag{87}$$

Identity (82) (or equivalently (83)), along with and (81), introduces the following theorem.

Theorem 10. ***Similarity structural equation***. *Any regular and smooth curve $\gamma = \gamma(\sigma)$ satisfies the following third order differential equation*

$$(\kappa\gamma'_\sigma)''_\sigma - \kappa\gamma'_\sigma = 0 \tag{88}$$

in similarity geometry where σ is the similarity arc length and κ is the Euclidean curvature.

Conjecture II. The unique planar geometry with an arc length ι in which each regular and smooth curve $\gamma = \gamma(\iota)$ is a solution to the the 3^{rd} ordinary differential equation

$$(\kappa\gamma'_\iota)''_\iota - \kappa\gamma'_\iota = 0 \tag{89}$$

where κ is the Euclidean curvature, is the similarity with $\iota = \sigma$.

In addition, (84) presents the following result.

Corollary 4. *In similarity geometry, the set $\{(\kappa\gamma'_\sigma)'_\sigma, (\kappa\gamma'_\sigma)''_\sigma\}$ forms the TN frame for a regular smooth convex curve γ - in other words*

$$\|(\kappa\gamma'_\sigma)'_\sigma\| = \|(\kappa\gamma'_\sigma)''_\sigma\| = 1, \quad (\kappa\gamma'_\sigma)'_\sigma \perp (\kappa\gamma'_\sigma)''_\sigma, \quad \textit{and} \quad (\kappa\gamma'_\sigma)^{(n+2)}_\sigma = (\kappa\gamma'_\sigma)^{(n)}_\sigma. \tag{90}$$

4.3 Structural equation in affine geometry

By (60)

$$\kappa^{1/3}\gamma'_\alpha = \mathrm{T}. \tag{91}$$

Hence

$$\begin{aligned} \|\kappa^{1/3}\gamma'_\alpha\| &= 1 \implies (\kappa^{1/3}\gamma'_\alpha)'_\alpha = \xi_1 \,\mathrm{N} \\ \implies \xi_1 &= \langle(\kappa^{1/3})'_\alpha\gamma'_\alpha + \kappa^{1/3}\gamma''_\alpha,\ \mathrm{N}\rangle = (\kappa^{1/3})'_\alpha\langle\gamma'_\alpha,\ \mathrm{N}\rangle + \kappa^{1/3}\langle\gamma''_\alpha,\ \mathrm{N}\rangle \implies \xi_1 = \kappa^{2/3}. \end{aligned}$$

Therefore

$$(\kappa^{1/3}\gamma'_\alpha)'_\alpha = \kappa^{2/3}\,\mathrm{N} \qquad \text{or} \qquad \kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)'_\alpha = \mathrm{N}, \tag{92}$$

which results in

$$\begin{aligned} &(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)'_\alpha)'_\alpha = \xi_2\,\mathrm{T} \implies \xi_2 = \langle(\kappa^{-2/3})'_\alpha(\kappa^{1/3}\gamma'_\alpha)'_\alpha + \kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)''_\alpha,\ \mathrm{T}\rangle \\ &\implies \xi_2 = (-\frac{4}{9}\kappa^{-7/3}\kappa'^{,-2}_\alpha + \frac{1}{3}\kappa^{-4/3}\gamma''_\alpha)\langle\gamma'_\alpha,\ \mathrm{T}\rangle + \kappa^{-1/3}\langle\gamma'''_\alpha,\ \mathrm{T}\rangle \implies \xi_2 = -\kappa^{2/3}. \end{aligned}$$

Hence

$$(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)'_\alpha)'_\alpha = -\kappa^{2/3}\,\mathrm{T} \qquad \text{or} \qquad -\kappa^{-2/3}(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)'_\alpha)'_\alpha = \mathrm{T}. \tag{93}$$

Continuing the same process gives

$$-\kappa^{-2/3}(\kappa^{-2/3}(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)')'_\alpha)'_\alpha = \mathrm{N}. \tag{94}$$

Identity (89) (or equivalently (90)), along with (88), results in the following theorem.

Theorem 11. ***Affine structural equation***. *Any regular smooth and curve $\gamma(\alpha)$ satisfies the following third order differential equation*

$$(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\alpha)'_\alpha)'_\alpha + \kappa\gamma'_\alpha = 0. \tag{95}$$

in affine geometry where α is the affine arc length and κ is the Euclidean curvature.

Conjecture III. The unique planar geometry with an arclength ι in which any regular and smooth curve $\gamma = \gamma(\iota)$ is a solution to the the 3^{rd} ordinary differential equation

$$(\kappa^{-2/3}(\kappa^{1/3}\gamma'_\iota)'_\iota)'_\iota + \kappa\gamma'_\iota = 0 \tag{96}$$

where κ is the Euclidean curvature, is the affine with $\iota = \alpha$.

Note, the coefficients in the structural equations (79), (85), and (92) are *merely* in terms of the Euclidean curvature and its derivatives in the visual geometry of interest.

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